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BUCKLING UNDER UNILATERAL CONSTRAINTS

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Abstract—The classic variational theory for eigenvalue problems is extended so as to define the Euler critical load of a unilaterally constrained beam. The Euler load is obtained by minimising a Rayleigh quotient on a convex subset of a Hilbert space. The variational formulation also provides a method of bounding the buckling load by comparison.

1. INTRODUCTION

In this paper we show how the linear theory of elastic stability, in particular the extremum properties of the critical load for a structure can be extended almost word for word to problems where, besides boundary conditions of classical type, perturbed displacements are subject to unilateral constraints.

A typical situation is that of the hinged beam, drawn in Fig. 1, under an axial compressive load *P*. The beam in the straight position is simply supported on platforms like *CC'*, *DD'*, which prevent transverse displacements of parts of the beam in one direction.

It is fairly obvious that the definition of critical state in this example is the immediate extension of the notion of buckling in the linear theory: i.e. the minimum axial load permitting non-trivial equilibrium configurations. The only difference is that now the admissible perturbed configurations must obey the constraints imposed by *CC'* and *DD'*.

On the other hand, it is not so obvious that the critical load can be determined as the minimum of a functional and that this minimum is effectively attained by an admissible function. It is known that the variational formulation of buckling problems is an effective tool for approximating or bounding the critical load, or simply for establishing its properties by comparison.

Many of the results of the classical variational formulation can be extended to treat buckling under unilateral constraints.

It is possible to show that the critical load is the minimum of a quadratic functional on a convex set of functions and that at least one minimizing function exists. These results are simple consequences of the calculus of variations on convex sets, which is an extension of the classical projection theorem in Hilbert space.

However, once the variational formulation of the buckling problem is understood, another question not yet fully explored is the dependence of the critical load upon the data of the problem and in particular upon the unilateral constraints. Results of this kind are well known in classical variational problems (see e.g. Courant–Hilbert[1] and Weinberger[2]), but the extension to variational problems with unilateral constraints is still incomplete[3]. These methods can provide information in three ways: comparison of the critical load with that of the corresponding problem without unilateral constraints; change of critical load under perturbation of the constraints; and optimal position of the unilateral constraints to maximise the critical load.

The variational formulation of elastic stability under constraints applies, of course, to beams, plates[4] and three-dimensional bodies. Here, however, we shall consider only Eulerian instability of the simple beam of Fig. 1. This simplifies the calculations without compromising generality and permits us to characterise quantitatively how the solution depends on the constraints.

The interesting fact is that many comparison properties of the buckling load previously accepted as conjectures can be now demonstrated as theorems.

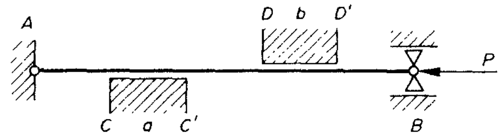

Fig. 1.

2. THE MINIMUM PROBLEM

We consider a hinged beam of length l and constant flexural rigidity EJ, which, in its fundamental straight configuration, is in contact with disjoint supports like CC', DD', ... If the beam is loaded by a force P, buckling will occur when other equilibrium configurations, like that shown dashed in Fig. 2, are possible.

In order to fit the problem into the correct framework, we introduce the Hilbert space $H^2(0, l)$, defined by closure with respect to the norm

$$\|u\|_2^2 = \int_0^l (u''^2 + u'^2 + u^2)\,dx, \tag{2.1}$$

of the set of functions $u \in \mathscr{C}^\infty[0, l]$.

We next consider the forms

$$a(u, u) = \int_0^l EJu''^2\,dx, \quad b(u, u) = \int_0^l u'^2\,dx, \tag{2.2}$$

which are quadratic functionals with domain $H^2(0, l)$.

In an analogous manner we can define the Hilbert space $H^1(0, l)$ and in particular the space $H_0^1(0, l)$, which is the closure of the class of functions $\mathscr{C}_0^\infty(0, l)$ with respect to the norm

$$\|u\|_1^2 = \int_0^l (u'^2 + u^2)\,dx.$$

We also define the subspace $V = H^2(0, l) \cap H_0^1(0, l)$.

We now consider the unilateral constraints CC', DD', which, if a certain orientation for u is chosen and $[\gamma, \gamma']$, $[\delta, \delta']$ are the positions of their end points, impose restrictions of the type

$$u \geq 0 \quad \text{on} \quad [\gamma, \gamma'], \quad u \leq 0 \quad \text{on} \quad [\delta, \delta']. \tag{2.3}$$

It is easy to show that (2.3) defines a convex subset K of V. In fact, if u_1 and u_2 are two functions satisfying (2.3) then $u = tu_1 + (1-t)u_2$ $(0 \leq t \leq 1)$ also satisfies (2.3) and therefore belongs to K.

After these preliminaries, we consider the ratio

$$P = \frac{a(u, u)}{b(u, u)}, \tag{2.4}$$

which we call the Rayleigh quotient. The critical load P_{cr} is the minimum value of the functional (2.4) for all functions u in the set K.

The essential point of the variational theory is to prove that the Rayleigh quotient has a minimum and that this value is actually attained for an admissible function $\bar{u}$. The existence of a

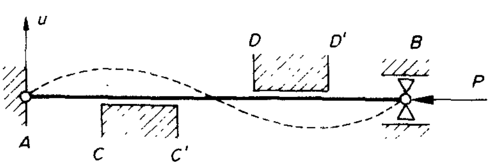

Fig. 2.

minimum is a consequence of the following properties (see e.g. Weinberger[2]):

1. $b(u, u)$ is a positive definite quadratic form.
2. $[a(u, u)]/[b(u, u)]$ is bounded below for all vectors u in K.
3. $b(u, u)$ is completely continuous with respect to $a(u, u)$, i.e. every sequence $\{u_n\}$ for which $a(u_n, u_n)$ is uniformly bounded contains a subsequence $\{u_{n'}\}$ such that

$$b(u_{n'} - u_{m'}, u_{n'} - u_{m'}) \to 0 \quad \text{as} \quad m', n' \to \infty.$$

Property 1 is an obvious consequence of definition (2.2) and the boundary conditions, which exclude the functions $u = \text{const}$. Property 2 derives from the Wirtinger inequality (see, e.g. Beckenbach and Bellman[5]), which, combined with the Schwarz inequality, gives

$$\int_0^l u^2\,dx \leq \beta \int_0^l u'^2\,dx = -\beta \int_0^l u''u\,dx \leq \beta\left(\int_0^l u''^2\,dx\right)^{1/2}\left(\int_0^l u^2\,dx\right)^{1/2}$$
$$\leq \frac{\beta}{\sqrt{(EJ)}}\left(\int_0^l EJu''^2\right)^{1/2}\left(\int_0^l u^2\,dx\right)^{1/2}, \tag{2.5}$$

where β is a positive constant.

Finally, in order to prove that $b(u, u)$ is completely continuous with respect to $a(u, u)$, we note that, for any two points x_1 and x_2 in $(0, l)$, we have

$$[v'(x_1) - v'(x_2)]^2 = \left(\int_{x_1}^{x_2} v''(x)\,dx\right)^2 \leq |x_2 - x_1| \int_{x_1}^{x_2} v''^2(x)\,dx.$$

If x_1, x_2 belong to an interval $[a, a+s]$ contained in $[0, l]$, and we take a double integral with respect to x_1 and x_2 over $[a, a+s]$, we obtain

$$2s\int_a^{a+s} v'^2(x)\,dx - 2\left(\int_a^{a+s} v'(x)\,dx\right)^2 \leq \int_a^{a+s} dx_1 \int_a^{a+s} dx_2|x_2 - x_1| \int_{x_1}^{x_2} v''^2(x)\,dx$$
$$\leq s^3 \int_a^{a+s} v''^2(x)\,dx. \tag{2.6}$$

We now divide $[0, l]$ into κ intervals I_i $(i = 1, \dots, \kappa)$ each of length $s = l/\kappa$. The inequality (2.6) does not depend on the origin of coordinates, so (2.6) holds over each interval. Since v in (2.6) is any function in K, we consider a uniformly bounded sequence $\{v_n\}$ such that $a(v_n, v_n) \leq c$ independently of n. Then (2.6) holds over each I_i for the difference $v_n - v_m$. Summing from 1 to κ we obtain

$$2s\int_0^l (v_n' - v_m')^2\,dx \leq 2\sum_{i=1}^{\kappa}\left(\int_{I_i}(v_n' - v_m')\,dx\right)^2 + s^3\int_0^l (v_n'' - v_m'')^2\,dx, \tag{2.7}$$

i.e.

$$b(v_n - v_m, v_n - v_m) \leq \frac{1}{s}\sum_{i=1}^{\kappa}\left(\int_{I_i}(v_n' - v_m')\,dx\right)^2 + \frac{s^2}{2EJ}a(v_n - v_m, v_n - v_m). \tag{2.8}$$

Since $a(v_n, v_n) \leq c$, by the triangle inequality we have $a(v_n - v_m, v_n - v_m) \leq 4c$. Consequently we can choose κ such that

$$\frac{2s^2c}{EJ} \leq \frac{\epsilon}{2}.$$

On the other hand, by the Schwarz inequality we know that

$$\left(\int_{I_i} v_n'\,dx\right)^2 \leq sb(v_n, v_n) \leq \frac{s\beta^2}{EJ}a(v_n, v_n).$$

This means that the sequences of real numbers $\int_{I_i} v_n' \, dx$ are uniformly bounded, and there exists a subsequence (still denoted by v_n') such that $\int_{I_i} v_n' \, dx$ converges to a limit a_i ($i = 1, \ldots, \kappa$). Thus, by the Cauchy criterion, there exists an N such that for $n, m > N$

$$\frac{1}{s}\sum_{i=1}^{\kappa}\left(\int_{I_i}(v_n' - v_m')\,dx\right)^2 \leq \frac{\epsilon}{2}. \tag{2.9}$$

Then (2.8) becomes

$$b(v_n - v_m, v_n - v_m) < \epsilon$$

which proves property (3).

Thus, there exists an element $u_1{}^K \in K$ for which the Rayleigh quotient assumes its minimum value. If K coincides with V, so that unilateral constraints are absent, the Rayleigh quotient must be minimised on the entire subspace V and it is easy to prove that

$$\min_{u \in V} \frac{a(u, u)}{b(u, u)} = \frac{\pi^2 EJ}{l^2}, \tag{2.10}$$

and hence P_{cr} is the Euler load for a simply supported beam with constant flexural rigidity. The minimum of the Layleigh quotient is also called an *eigenvalue*, and the (not necessarily unique) minimizing element u_1 is said to be an *eigenvector*.

2.1 *Remark*

The Rayleigh ratio is a homogenous function of degree zero on u. Therefore its value does not change when u is replaced by κu where κ is any constant. Moreover, when K is defined according to (2.3), if u is an admissible function of the minimum problem, then cu, with $c \geq 0$, is also admissible. This means that K is a convex cone with vertex at the origin. Then if u is a solution to the minimum problem so also is cu, with $c \geq 0$.

3. COMPARISON PROPERTIES OF THE CRITICAL LOAD

The interest in using a comparison theorem for the critical load stems from the circumstance that, while it is usually difficult to evaluate the minimum of the Rayleigh quotient for $u \in K$, there exist many ways of calculating or approximating the minimum for $u \in V$.

More generally, suppose that we want to compare the critical load of a beam under certain constraints with the critical load of the same beam in which some constraints are removed. In the first case we must seek the minimum of the Rayleigh quotient on K, in the second case we must minimize the Rayleigh quotient on a set $K \subset K'$. It is obvious that the first minimum is not less than the second and we can thus state the following theorem:

Theorem 1. *The critical load of a unilaterally constrained beam does not increase with the removal of the unilateral constraints.*

This theorem describes the dependence of the critical load on the set where the minimum is sought. Another theorem characterises how the critical load depends on $a(u, u)$ and $b(u, u)$. Suppose in fact that $a(u, u)$ and $b(u, u)$ are altered to $a'(u, u) \geq a(u, u)$ and $b'(u, u) \leq b(u, u)$. If

$$P_{cr}' = \min_{u \in V} \frac{a'(u, u)}{b'(u, u)}, \tag{3.1}$$

then clearly $P_{cr}' \geq P_{cr}$. This implies the following:

Theorem 2. *Increasing* a(u, u) *and decreasing* b(u, u) *will not decrease the critical load.*

The theorems above are simple extensions of a classical monotonicity principle† for eigenvalues and apply in general. We may now ask which further properties of the solutions

†This monotonicity principle was first enunciated by Courant for linear eigenvalue problems.

derive from the particular form of the Rayleigh quotient and the set K as specified in Remark 2.1.

In this case if we represent V as a plane, the convex cone K can be drawn as a plane sector with vertex at the origin (Fig. 3). If we minimise the Rayleigh quotient on V (that is, neglecting unilateral constraints) we are able to find at least one element $u_1 \in V$ at which the minimum is attained. Since also κu, with κ a parameter, is a solution, the Rayleigh quotient has its minimum along the entire straight line κu_1. We denote by λ_1 this minimum, which, as a consequence of the assumptions on $a(u, u)$ and $b(u, u)$, is real and positive, so that the Euler load P_{cr} is exactly λ_1, and u_1 the corresponding eigenfunction. For our purposes it is also useful to introduce the hither eigenvalues and eigenvectors of the Rayleigh quotient defined recursively as follows (see, e.g. Weinberger [2])

$$\lambda_\kappa = \min_{u \in V} \frac{a(u, u)}{b(u, u)}, \tag{3.2}$$

with the conditions $b(u, u_1) = b(u, u_2) = \ldots = b(u, u_{\kappa-1}) = 0$. If we now minimize the Rayleigh quotient on K and denote by $u_1{}^K$ the corresponding solution, then $cu_1{}^K$ $(c \geq 0)$ is also a solution. Therefore the Rayleigh quotient has its minimum along the entire ray $cu_1{}^K$ (Fig. 3). Let $\mu_1 = P_{cr}^K$ be this minimum.

The geometrical situation of Fig. 3 suggests how to compare μ_1 with λ_1.

We may distinguish the two cases, when u_1 is and is not contained in K. It is clear that, if $u_1 \in K$, the minimum of the Rayleigh quotient on K coincides with the minimum on V, and $P_{cr}^K = P_{cr}$. In this case the unilateral constraints do not alter the Euler critical load. When u_1 is exterior to K, then $u_1{}^K$ lies on the boundary of K. In this case we may denote by

$$\theta = \cos^{-1} \frac{b(u_1, u_1{}^K)}{\sqrt{[b(u_1, u_1) b(u_1{}^K, u_1{}^K)]}} \tag{3.3}$$

the angle that κu_1 forms with $cu_1{}^K$ (Fig. 3).

The proof is by contradiction. Suppose in fact that $u_1{}^K$ is interior to K. Then the convex combination

$$u_1{}^t = t u_1 + (1-t) u_1{}^K \quad (0 \leq t \leq 1)$$

is still an element of K for t sufficiently small. But in this case it is easy to verify that

$$P_{cr}^t = \frac{a(u_1{}^t, u_1{}^t)}{b(u_1{}^t, u_1{}^t)} = \frac{a(u_1{}^K + u_1{}^t - u_1{}^K, u_1{}^K + u_1{}^t - u_1{}^K)}{b(u_1{}^K + u_1{}^t - u_1{}^K, u_1{}^K + u_1{}^t - u_1{}^K)} = \frac{a(u_1{}^K, u_1{}^K)}{b(u_1{}^K, u_1{}^K)}$$
$$+ \frac{t^2}{b(u_1{}^t, u_1{}^t)} [a(u_1 - u_1{}^K, u_1 - u_1{}^K) - P_{cr}^K b(u_1 - u_1{}^K, u_1 - u_1{}^K)] \leq \frac{a(u_1{}^K, u_1{}^K)}{b(u_1{}^K, u_1{}^K)} = P_{cr}^K, \tag{3.4}$$

since, by Theorem 1, we know that

$$\frac{a(u_1{}^K, u_1{}^K)}{b(u_1{}^K, u_1{}^K)} \geq \frac{a(u_1, u_1)}{b(u_1, u_1)}.$$

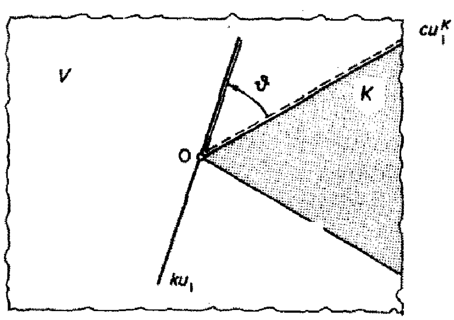

Fig. 3.

Thus (3.4) would mean that $u_1{}^K$ is not a point of minimum for the Rayleigh quotient.

We have then proved

Theorem 3. *If* u_1 *belongs to* K *the critical load coincides with the Euler load*; *if* u_1 *lies outside* K *the* (*necessarily greater*) *critical load is attained on a ray of the boundary of* K.

An example of the first type is given by the beam of Fig. 4(a) where the critical load is clearly the Euler load (2.10) and any perturbation of u_1 is admissible. Figure 4(b) illustrates a situation of the second kind, where the solution $u_1{}^K$ cannot be arbitrarily perturbed without affecting the conditions of admissibility.

Another useful comparison property can be derived from a knowledge of the higher eigenvalues of the Rayleigh quotient defined by (3.2).

It may happen when u_1 is exterior to K, that one of the higher eigenfunctions, for instance u_n, belongs to K, while u_{n-1} is exterior to it. Since u_{n-1} and u_n are two relative minima of the Rayleigh quotient satisfying the condition $\lambda_{n-1} \leq \lambda_n$, on using the technique of the proof of Theorem 3 we obtain that

$$\lambda_{n-1} \leq P_{cr}^K \leq \lambda_n. \tag{3.5}$$

We may thus prove the following theorem:

Theorem 4. *If* u_n *is the first eigenfunction of the unconstrained problem compatible with the unilateral constraints, then the critical load is bounded from below by* λ_{n-1} *and from above by* λ_n.

As an application of the theorem, we consider the beam of Fig. 5 with unilateral supports offering reactions of alternate sign on three intervals of length $l/3$. In this case, we easily see that the first admissible eigenfunction is $\sin(3\pi/l)z$ and consequently

$$\frac{4\pi^2 EJ}{l^2} = \lambda_2 \leq P_{cr}^K \leq \lambda_3 = \frac{9\pi^2 EJ}{l^2}.$$

3.1 *Remark*

Observe that, when $u_1{}^K$ belongs to the boundary of K, the solution necessarily touches the constraints in some points.

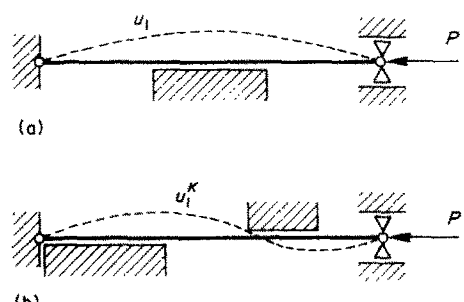

Fig. 4.

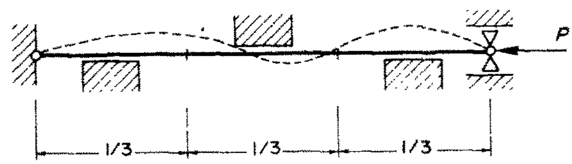

Fig. 5.

4. INFLUENCE OF THE CONSTRAINTS

We have so far considered comparison properties of the critical load with the eigenvalues of the corresponding unconstrained problem.

We wish now to investigate more closely how the critical load depends on the shape of K and on its perturbations.

The first result of this kind is based on the fact that both $a(u, u)$ and $b(u, u)$ are quadratic functionals on u and, consequently, the Rayleigh quotient does not change if u is replaced by $-u$.

Thus, if u_1^K is a solution of the minimum problem under certain constraints, $-u_1^K$ is not in general admissible (see Section 2.1). However, $-u_1^K$ becomes admissible if the unilateral constraints are symmetrically reversed with respect to the axis of the beam. This proves

Theorem 5. *The critical load does not change after symmetrical reversal of the unilateral constraints.*

For instance the critical load of the beam in Fig. 6 does not change if the unilateral supports a and b are replaced by a' and b'.

A further consequence of Theorem 3 is that the solution is partially indifferent to the variations of K.† Two situations occur. When u_1 is interior to K, and therefore $u_1^K = u_1$, the minimum of the Rayleigh quotient is attained in the interior of K and coincides with the absolute minimum on V. It follows that we can arbitrarily alter K, without changing the solution, provided that K continues to contain u_1. When u_1 is exterior or lies on the boundary of K the minimum of the Rayleigh quotient is attained along one of the generators of K. We can thus alter the remaining part of K without changing the solution, provided that the ray containing u_1^K remains the point of minimum for the Rayleigh quotient (2.4). This result can be summarized by the following:

Theorem 6. *Once* u_1^K *is known, the solution is indifferent to those perturbations of* K *preserving the character of a minimum for* u_1^K.

The practical consequences of Theorem 6 can be illustrated by the following examples. In the beam of Fig. 7(a) the unilateral support a can be arbitrarily translated without modifying the critical load (which is the Euler load). In the beam of Fig. 7(b) the buckled configuration u_1^K does not change if one adds another support c in the region where the displacement is directed downwards.

Theorem 4 can be also used to solve problems of optimum design. Suppose that a certain number of unilateral constraints are given, of which the dimensions are prescribed, but not the positions along the span of the beam. We may ask what disposition of the constraints minimizes the critical load. A shift of the supports causes a perturbation of K, and we wish to find the shape of K, compatible with the dimensions of the constraints, that maximizes the Rayleigh quotient.

A criterion of optimal disposition of a given set of constraints is made precise by

Theorem 7. *If, by changing the position of the unilateral constraints,* K *is never empty, the perturbation of* K *which maximizes the critical load is that excluding the highest possible number of eigenfunctions* $\{u_n\}$ *in increasing order.*

Proof. The proof is a simple consequence of two properties, the monotone ordering of the eigenvalues of the Rayleigh quotient and the completeness of the system $\{u_n\}$. Since K is not empty and $\{u_n\}$ is complete, some linear combinations of u_n must necessarily belong to K. Since the eigenvalues corresponding to $\{u_n\}$ are not decreasing with n, the best perturbation of K is that excluding $u_1, u_2, \ldots$ as far as the eigenfunction of highest order allowed.

The following example clarifies the situation. If K consists of two unilateral point supports a, b (Fig. 8), we may ask what is the position of a, b maximizing the critical load, By virtue of Theorem 4 it is clear that

$$\lambda_1 \leq P_{cr}^K \leq \lambda_2. \tag{4.1}$$

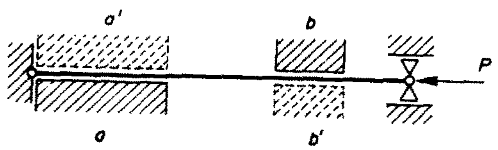

Fig. 6.

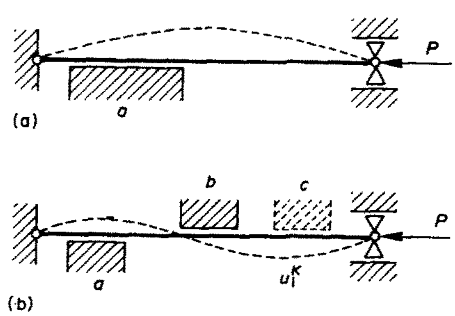

Fig. 7.

†This property was exploited by Mancino and Stampacchia[6].

In fact, if a and b lie on the same part of the geometrical axis, $u_1 = \sin(\pi z)/l$ is admissible and P^K_{cr} is λ_1, the Euler load. If a and b are placed on opposite sides of the geometrical axis the critical load satisfies (4.1). When a and b are placed at the mid point, P^K_{cr} is λ_2, and reaches its maximum value.

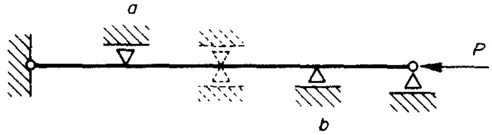

Fig. 8.

5. THE SIMPLE BEAM ON POINT SUPPORTS

We have so far derived certain comparison properties of the critical load which apply, not only to simple beams, but to more complex elastic structures. However, for a simple beam with unilateral point supports we may obtain further properties of the critical load and the buckled configuration. In some cases we are even able to find the exact solution.

Let us consider, for simplicity, a beam of constant flexural rigidity $EJ = 1$, length $l = \pi$, with hinged ends and two unilateral point supports at distances $\alpha_1\pi$ and $\alpha_2\pi$ ($0 < \alpha_1 < 1/2$, $1/2 < \alpha_2 < 1$) from one end (Fig. 9). Under the action of the axial load P the possible buckled states are those minimizing the Rayleigh quotient

$$\frac{a(u, u)}{b(u, u)} = \frac{\int_0^\pi u''^2\,dz}{\int_0^\pi u'^2\,dz}, \tag{5.1}$$

among all functions $u(z)$ with piecewise continuous second derivatives such that $u(0) = u(\pi) = 0$ and moreover $u(\alpha_1\pi) \leq 0$, $u(\alpha_2\pi) \geq 0$. These two last inequalities define the convex set K in our problem.

According to the properties of Theorem 3, we know that, if the absolute minimum of (5.1) is not interior to K, it falls on the boundary of K, which implies that the solution must vanish at least at one of the points $\alpha_1\pi$, $\alpha_2\pi$. Since these points are finite in number, it is clear that the solution must be sought among three possible schemes: a continuous beam with two intermediate supports at $\alpha_1\pi$ and $\alpha_2\pi$ [Fig. 9(a)]; a continuous beam with one intermediate support at $\alpha_1\pi$ or $\alpha_2\pi$ [Fig. 9(b) and (c)]. The system (a), however, gives a critical load necessarily not less than (b) and (c), because it imposes two zeros on the admissible functions instead of one. It follows that the critical load is the minimum between the critical loads of two continuous beams with one intermediate support at $\alpha_1\pi$ or $\alpha_2\pi$. But the critical load of a beam on three supports with the central support at distance $\alpha_1\pi$ or $(1-\alpha_2)\pi$ from the closest end is

$$P_{cr} = \frac{\kappa}{(1-\alpha_1)^2}\ \left(\text{or}\ \frac{\kappa}{\alpha_2^2}\right), \tag{5.2}$$

where κ is a computable factor.† Thus the critical load is

$$P^K_{cr} = \min\left\{\frac{\kappa}{(1-\alpha_1)^2}, \frac{\kappa}{\alpha_2^2}\right\}. \tag{5.3}$$

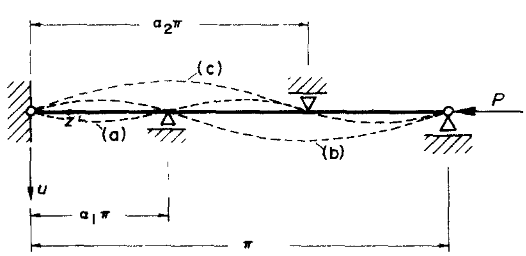

Fig. 9.

†The factor κ is the lowest zero of a trascendental equation called Berry's equation (see Timoshenko and Gere[7]).

It is interesting to observe that, though the critical load P_{cr}^K is unique, the minimising solution is not necessarily unique. For instance, if $\alpha_1 = (1 - \alpha_2)$ there are at least two buckled states, (b) and (c), admitting the same critical load.

This argument can of course, be used to evaluate the critical load of a beam on an arbitrary number of unilateral point supports.

In general, the true critical load is attained for that equilibrium configuration which excludes the maximum number of intermediate supports.

5.1. *Example.* If in the beam of Fig. 9 we take $\alpha_1 = (1 - \alpha_2) = 1/3\pi$, the critical load is given by (Section 2.5, [7])

$$P_{cr}^K = \frac{14.9}{(2/3\pi)^2} \sim 3.40.$$

On the other hand, if we consider the admissible function $v = \sin 2z$, the corresponding value of the Rayleigh quotient is

$$\frac{a(v,v)}{b(v,v)} = \frac{4^2 \int_0^\pi \sin^2 2z \, dz}{4 \int_0^\pi \cos^2 2z \, dz} = 4,$$

and this upper bound for P_{cr}^K agrees with Theorem 4.

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